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Crack Propagation in a Linearly Viscoelastic Strip

The tip velocity of a crack propagating through a viscoelastic material depends on geometry, applied load and its history, and material properties. A consideration of the work done by the unloading tractions at the crack tip shows that, for a large crack propagating through an infinitely long strip under constant lateral strain, the rate of propagation can be calculated from a knowledge of the intrinsic fracture energy (a material constant), the material creep compliance, and an additional size parameter. This parameter vanishes from the analysis if the material is elastic, and the familiar instability criterion is obtained in this case. Comparison with experimental data is provided and the consequences of step loadings are examined.

Introduction

Even though there are two apparently different approaches to brittle fracture, the instability behavior of cracks in linearly elastic solids is well understood. On the one hand, the application of the first law of thermodynamics to the problem of a growing crack [1][2] associated with a singular stress field at its tip leads to the now classical instability criterion. On the other hand, the quasi-atomistic approach of the so-called equilibrium crack [2] leads to an identical result through use of a nonsingular stress field representation. The equivalence of the two approaches is based essentially on the equivalence of the work done by the unloading tractions at the tip of an advancing crack, wherein the precise distribution of the stresses at the crack tip plays a secondary role. Fundamentally, it was Irwin's [3] demonstration of the local nature of the fracture process that elucidated this connection, as well as the supplemental investigations of Bueckner [4] and Sanders [5]. As a consequence of these expositions, fracture in brittle solids has become looked upon as a local phenomenon rather than a global one as in [1].

The continuum mechanical description of crack growth in materials other than linearly elastic ones is not as well understood, the primary reason being the lack of simple mathematical tools for an analytic description of the deformations in yielding materials under crack growth. While it would lead too far from our present objective to review even the most important work in metal fracture, suffice it to state that work on a crack growth criterion for very ductile materials is in progress [6–9] but it does not seem promising that a general criterion for yielding metals will be found. Partial solutions such as the concept of quasi-brittle failure advanced by Orowan [10] and Irwin [11] are useful and lean heavily on the principles of linear fracture mechanics, which permeate almost all of the work on fracture in nonlinear solids.

With the exception of fatigue and creep fracture [12], metal failure is hardly rate-dependent. In contrast, the failure of organic glasses and other polymeric solids exhibit strong rate effects which complicate the understanding of the fracture process. Inasmuch as linear fracture mechanics has illuminated the failure process in nonlinear, rate-insensitive materials, it seems prudent to investigate first the problem of crack propagation in a linearly viscoelastic solid.

Because the stress-strain analysis of a viscoelastic solid under time-varying surface tractions such as encountered in a moving crack is, in general, not readily performed, the global energy balance [1] cannot be carried out. Consequently, there is little information concerning the effect of viscoelastic properties on the process of crack growth. Starting from the concept of a maximum strain sustained by a viscoelastic solid at the crack tip, Williams [13] pointed out that a crack grows exponentially in a sheet of a Voigt material. This result was also shown to hold for the antiplane shear case by McClintock [14]. The onset of crack propagation through a bubble geometry under hydrostatic tension was studied by Williams [15]. Wnuk and Knauss [16] examined the actual case of a penny-shaped crack in a linearly viscoelastic solid exhibiting a deformation-rate-sensitive yield stress. These investigations are primarily of a

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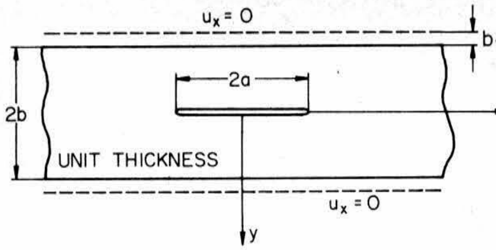

Fig. 1 Strip geometry

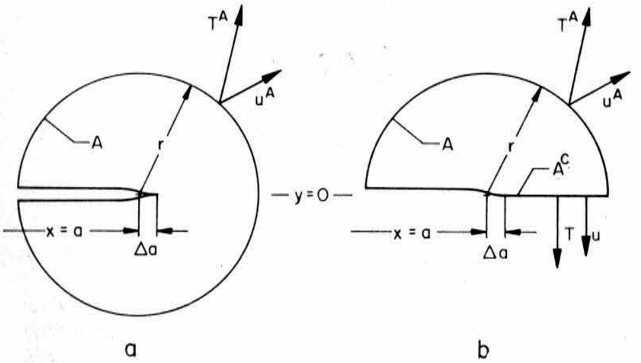

Fig. 2 Control surface around crack tip

qualitative nature because either the material representation or the geometry is overly simplified.

It is the purpose of this paper to derive a crack-propagation model based on the first law of thermodynamics and to examine its usefulness in application to a viscoelastic solid. Inasmuch as comparison of a theory and its experimental evaluation requires a realistic representation, the following work suffers from the assumption of linear viscoelastic material behavior whereas the material in the immediate vicinity of the crack tip is under large strain and clearly does not behave in a linear fashion. Nevertheless, if close agreement between theory and experiment occurs despite this discrepancy, we may have resolved a problem of some practical importance.

Because any time variation in the boundary conditions complicates the viscoelastic analysis, it is advantageous in an initial investigation to consider the simplest possible situation. Such a situation is provided by the steady growth of a large crack $a > 1.5b$, Fig. 1, along the center line of an infinitely long strip under constant lateral strain ϵ_0. The only variables entering the isothermal problem are then the strain ϵ_0 and, in dependence on ϵ_0, the velocity of crack growth v.

Derivation of Power Equation

Consider the tip of a traction-free crack along $y = 0$ to be surrounded by a control surface A, as shown in Fig. 2(a), for some time t. The crack propagates through a thin plate of constant thickness and extends from one plate face to the other. A state of plane stress is assumed to exist in the plate. The rate of work done by the tractions T_i^A acting on A is

$$\dot{W} = \int_A T_i^A \dot{u}_i^A ds \tag{1}$$

This quantity is equal to the rate of energy dissipation $\dot{D}_v$, the rate of increase of surface energy $\dot{D}_s$, and the rate of change of reversibly stored energy $\dot{E}$ in the control volume. The plate temperature during crack propagation is assumed to be constant, and other energy contributions like kinetic energy and heat energy are neglected. This investigation restricts itself to small enough crack velocities to justify these assumptions. With dots denoting time derivatives, the power equation for the control volume thus reads

$$\dot{W} = \dot{E} + \dot{D}_v + \dot{D}_s \tag{2}$$

Limiting ourselves to plate geometries and external loadings that are symmetrical over the x-axis, we may consider the crack to propagate along a straight line identical to the x-axis. Suppose now that the lower half of the control volume is replaced by the forces it exerts on the upper half, and denote these forces by $T_i(x, t)$ as in Fig. 2(b). Because of the symmetry of the problem under consideration, the forces $T_i(x, t)$ are normal to the x-axis. Since all other forces acting on the control surface remain unchanged, the power equation for the upper half of the control volume simply reads

$$\frac{1}{2}\dot{W} + \int_{a-r}^{a+r} T_i(x, t)\dot{u}_i(x, t)dx = \frac{1}{2}(\dot{D}_v + \dot{E}) \tag{3}$$

where $u_i(x, t)$ denotes the displacement along the x-axis. A comparison of equations (2) and (3) leads to the simplified statement of energy conservation

$$-2\int_{a-r}^{a+r} T_i(x, t)\dot{u}_i(x, t)dx = \dot{D}_s \tag{4}$$

Remembering that the crack surface is free of tractions for $x < a$, and admitting nonzero displacements $u_i(x, t)$ a small distance Δa ahead of $x = a$, we may write equation (4) as

$$-2\int_{a}^{a+\Delta a} T_i(x, t)\dot{u}_i(x, t)dx = \dot{D}_s \tag{5}$$

Geometrically, the crack tip is thus located at $x = a + \Delta a$. The condition of a traction-free crack surface will here be used as definition for the crack length, and $x = a$ will henceforth be referred to as the location of the crack tip.

We shall later demonstrate that the energy required to form a unit of new surface can be considered a constant, say, S. The rate of increase of surface energy is therefore $\dot{D}_s = 2Sv$, where the term $v = \dot{a}$ is the crack-tip velocity and the factor 2 accounts for the creation of two fracture surfaces. The power equation can now be written as

$$-\int_{a}^{a+\Delta a} T_i(x, t)\dot{u}_i(x, t)dx = Sv \tag{6}$$

The physical meaning of the quantity Δa will become clearer during the following development.

Simplification of Power Equation

We remark parenthetically that the derivation of (6) implies continuous, nonsingular tractions $T_i(x, t)$ as well as continuous displacement gradients along the crack axis as proposed by Barenblatt [2]. Primarily, for reasons of simplicity, however, we should like to employ the singular stress distribution, which is obtained when no modifications near the crack tip are introduced. For this purpose we approximate the continuous crack propagation by a stepwise process, allowing the crack to propagate in small jumps of constant length Δa, with $a \gg \Delta a$. This process is illustrated in Fig. 3. Solid lines correspond to the stress distributions and displacements at some time t_j, and the broken curves correspond to successively later times $t = t_j + \tau < t_j + \Delta t$. The curves identified by $\tau = \Delta t$ represent the stresses and displacements at the end of the current jump and the beginning of the subsequent jump at $t = t_j + \Delta t$.

During each jump, the tractions over $a \leq x < a + \Delta a$ decrease from their maximum value to zero while the crack opening increases from zero to its maximum value, depending on the jump duration Δt. The work done during unloading of the tractions is easily calculated by considering the tractions and correspond-

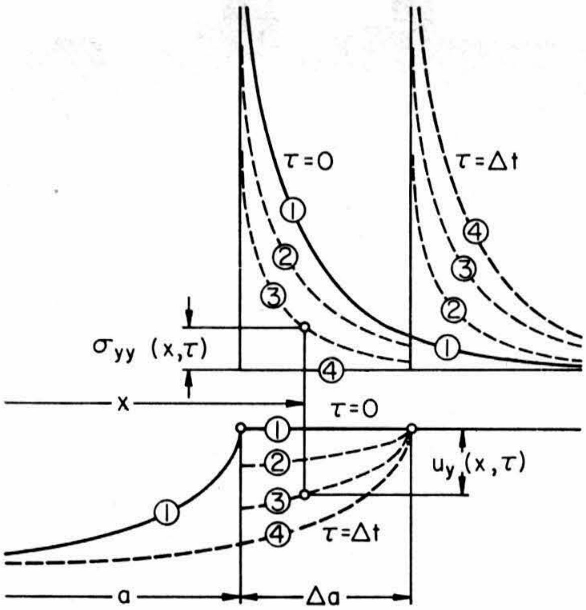

Fig. 3 Crack opening

ing displacements. The rate of crack propagation v at some time t_j can be calculated from the jump duration at this time and is given by $v = \dfrac{\Delta a}{\Delta t}$. With this interpretation in mind, we write equation (6) as

$$-\frac{1}{\Delta t}\int_a^{a+\Delta a}\int_{t_j}^{t_j+\Delta t}\sigma_{yy}(x, a, \tau - t_j)\dot{u}_y \times (x, a + \Delta a, \tau - t_j)d\tau dx = S\frac{\Delta a}{\Delta t} \quad (7)$$

where σ_{yy} and u_y stand for the normal stresses and displacements along the crack axis as they are obtained from the solution for a thin plate that is symmetrical over the x-axis and contains a line crack whose tip is located at $x = a$ and $x = a + \Delta a$, respectively.

Evaluation of Energy-Release Rate

The left-hand side of the simplified power equation (7) can be looked upon as the rate at which energy is released during a small extension of the crack. To evaluate this energy-release rate we need to know the stresses σ_{yy} and their rate of decrease, and the normal displacements in the small interval $a \leqslant x < a + \Delta a$. Consider the crack at some time t_j to be extended by Δa but held closed by appropriate tractions $\sigma_{yy}^*(x, a, t_j)$. As indicated in Fig. 3, we now allow these tractions to decrease until they vanish at time $t = t_j + \Delta t$, i.e.,

$$\sigma_{yy}(x, a, t - t_j) = \left[1 - \frac{t - t_j}{\Delta t}\right]\sigma_{yy}^*(x, a, t_j), \qquad \begin{array}{l} t_j \leqslant t \leqslant t_j + \Delta t, \\ a \leqslant x < a + \Delta a. \end{array} \quad (8)$$

The x-dependence of this stress is assumed to remain unchanged with time. The choice of a constant rate of unloading of the tractions σ_{yy}^* is somewhat arbitrary and has primarily been made for the purpose of simplicity. Another continuous unloading rate would give rise to a slightly different, but from a practical viewpoint, indiscernible end result [17]. The time which the crack needs to complete the jump from $x = a$ to $a + \Delta a$ is again denoted by Δt.

With the following definition for a nondimensional stress-intensity factor,

$$I_n(a) = \lim_{x\to a}\sqrt{\frac{x - a}{b}}\,\frac{\sigma_{yy}^*(x, a, t)}{\hat{\sigma}_{yy}(t)} \quad (9)$$

the stress σ_{yy}^* in the immediate vicinity of the crack tip can be represented as

$$\sigma_{yy}^*(x, a, t) = \sqrt{\frac{b}{x - a}}\,I_n(a)\hat{\sigma}_{yy}(t), \qquad \frac{x - a}{a} \ll 1 \quad (10)$$

where $\hat{\sigma}_{yy}$ denotes the stress σ_{yy} which would exist in the plate if the crack was absent.

The displacements u_y in a linearly viscoelastic material with constant Poisson's ratio ν due to the stress history given by (8) are found to be equal to

$$u_y(x, a + \Delta a, t - t_j) = -4bI_n(a + \Delta a)\sqrt{\frac{a + \Delta a - x}{b}}\,\frac{\hat{\sigma}_{yy}(t_j)}{\Delta t}\int_{t_j}^{t} D_{cr}(\tau - t_j)d\tau,$$
$$t_j \leqslant t \leqslant t_j + \Delta t, \qquad \frac{a + \Delta a - x}{a + \Delta a} \ll 1 \quad (11)$$

where $D_{cr}(t)$ denotes the tensile creep compliance of the material [18] and the time t_j marks the beginning of the step under consideration.

The left-hand side of equation (7) is now easily evaluated by substitution of expressions (8), (10), and (11) and leads to the following expression for some time t_j:

$$2\pi bI_n^2(a)\hat{\sigma}_{yy}^2(t_j)\left[D^{(1)}\left(\frac{\Delta a}{v}\right) - \frac{1}{2}D^{(2)}\left(\frac{\Delta a}{v}\right)\right] = S, \qquad v = \frac{\Delta a}{\Delta t} \quad (12)$$

where $I_n(a + \Delta a)$ has been approximated by $I_n(a)$ in view of $\Delta a \ll a$, and the quantities $D^{(n)}\left(\dfrac{\Delta a}{v}\right)$ are time-weighted averages of the creep compliance defined by

$$D^{(n)}(t)\begin{cases} = \dfrac{n}{t^n}\displaystyle\int_0^t \tau^{n-1}D_{cr}(\tau)d\tau & n = 1, 2 \\ = D_{cr}(t) & n = 0 \end{cases} \quad (13)$$

It should be noted that, for vanishing as well as for infinite argument of these functions, the bracket in equation (12) reduces to $\frac{1}{2}D_{cr}(0)$ and $\frac{1}{2}D_{cr}(\infty)$, respectively.

Equation (12) and other relationships to be derived from it can be simplified by introducing the function

$$G(t) = 2[D^{(1)}(t) - \tfrac{1}{2}D^{(2)}(t)] \quad (14)$$

Crack Propagation in a Strip

Consider the crack geometry shown in Fig. 1. The clamped boundaries are displaced normal to the crack so as to produce a constant and uniform lateral strain ϵ_0 far ahead of the crack tip. In this region, which is undisturbed by the presence of the crack, the strip material is furthermore assumed to be in its relaxed state. The stress $\hat{\sigma}_{yy}(t_j)$ is a constant in this case and is given by

$$\hat{\sigma}_{yy} = \frac{E_r\epsilon_0}{1 - \nu^2} \quad (15)$$

with E_r denoting the long-time, or relaxation, modulus of the material. Provided the crack length a is greater than $1.5b$, the stress-intensity factor becomes independent of crack length [18, 19] and assumes the constant (nondimensional) value

$$I_n = \sqrt{\frac{1 - \nu^2}{2\pi}} \quad (16)$$

For an incompressible material the power equation (12) thus reduces to a simple equation relating the strain ϵ_0 and the crack velocity v; namely,

$$\frac{2}{3} b\epsilon_0{}^2 E_r{}^2 G\left(\frac{\Delta a}{v}\right) = S \tag{17}$$

This result may be generalized to incorporate the effect of temperature by making use of the classical theory of rubber elasticity [20] and by assuming the material to be thermorheologically simple [21]. With reference to 0 deg C, equation (17) then reads

$$\frac{2}{3} b\epsilon_0{}^2 E_r{}^2 \frac{T}{273} G\left(\frac{\Delta a}{v\phi_T}\right) = S \tag{18}$$

where the time-temperature shift factor is denoted by ϕ_T.

Before we compare the relationship established in (18) with experimental data, it seems appropriate to comment on some limit cases. We note first that as $v \rightarrow 0$ the function $G\left(\frac{\Delta a}{v\phi_T}\right)$ approaches its maximum value $D_{cr}(\infty) = \frac{1}{E_r}$. In this case the strain ϵ_0 tends toward a lower limit

$$\epsilon_{0\min} = \sqrt{\frac{3S}{2bE_r} \frac{273}{T}} \tag{19}$$

No crack propagation is possible if ϵ_0 is smaller than this limiting value. If, on the other hand, $v \rightarrow \infty$ the function $G\left(\frac{\Delta a}{va_T}\right)$ tends to its lower limit $D_{cr}(0) = \frac{1}{E_g}$, where E_g denotes the short-time, or glassy, modulus. Therefore, if ϵ_0 exceeds the upper limit,

$$\epsilon_{0\max} = \sqrt{\frac{3SE_g}{2bE_r{}^2} \frac{273}{T}} = \sqrt{\frac{E_g}{E_r}}\, \epsilon_{0\min} \tag{20}$$

the crack-propagation process is governed by wave mechanics only, and (18) is inapplicable. Since E_g/E_r is on the order of 10^2–10^3 for many polymers, the upper limit on ϵ_0 is about 10–30 times as large as the lower limit.

In case the material is elastic with a Young's modulus E, the function $G(t)$ reduces to the constant $1/E$. The classical instability criterion for a strip with central crack is then regained

$$\epsilon_{0cr} = \sqrt{\frac{3S}{2bE}}, \qquad \frac{a}{b} > 1.5 \tag{21}$$

and the length Δa disappears in the end result in accordance with the work of Irwin [3] and others.

Comparison With Experiment

The polyurethane elastomer Solithane 113 [22] served as test material for the comparison of theory and experiment. The composition used for these tests was made from equal volumes of resin and catalyst and is referred to as Solithane 50/50. The function $G(t)$ for this material is shown in Fig. 4, together with the reciprocal uniaxial relaxation modulus $E_{rel}{}^{-1}(t)$ and the creep compliance $D_{cr}(t)$. The function $G(t)$ was calculated from $D_{cr}(t)$, which, in turn, had been calculated from the experimentally determined relaxation function [22]. The rubbery modulus E_r of this material is $E_r = 430$ psi at 0 deg C.

We have not yet commented on the physical significance of the jump size Δa, which does not vanish in general from the crack propagation equation (18); nor on the meaning of the intrinsic fracture energy S. It should be pointed out again that we consider S as a probably temperature-dependent but rate-insensitive material property. It is the lower limit of what is often called the tear energy [23]. The determination of S by means of a crack-propagation experiment requires the reduction of the energy dissipation which occurs in the process to as small a value as possible. This can be accomplished by measuring S for very small crack velocities at temperatures well above the glass transition temperature or by swelling the material in a suitable solvent and testing it in this state [22, 24]. Both methods were applied to determine the intrinsic fracture energy of Solithane 50/50 and led to the value $S = 0.1$ lb/in. ± 20 percent. A temperature dependence of S could not be detected in the tested range from −5 deg C to 50 deg C.

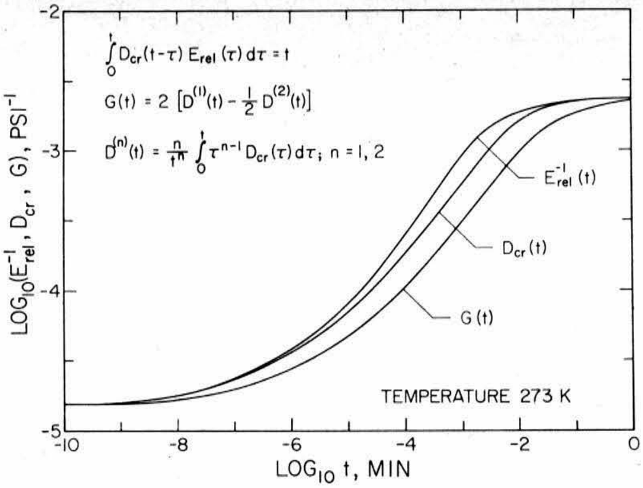

Fig. 4 Relaxation function, creep function, and G function for Solithane 50/50

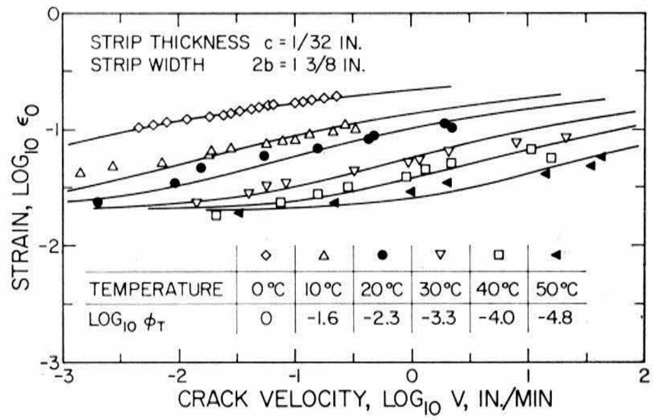

Fig. 5 Experimental and theoretical relationship among strain, temperature, and crack velocity in a strip of Solithane 50/50

The only unknown is now Δa. Since a change in Δa amounts to a shift of the theoretical strain-versus-crack-velocity curve, the jump size can easily be determined by matching this curve with one or several experimental points. In this manner Δa for Solithane 50/50 was found to be $\Delta a = 134$ Å.

The small size of Δa indicates that probably it is not purely a quantity of continuum mechanics, nor, however, of clearly molecular significance. Williams [13] and Bueche and Halpin [25] modeled viscoelastic crack propagation by assuming polymer strands to break successively at the crack tip. The width of these strands was suggested to be between 1 and 100 Å [25]. Although Δa is possibly of the same order of magnitude as the thickness of a strand, the criterion of a limiting strain or stress at the crack tip [13, 25] leads to fundamentally different results from those obtained in this work. The presence of a length Δa in the crack-propagation equation is not solely the consequence of assuming the crack to propagate in a stepwise manner. If a continuous process had been considered with the help of a stress distribution as suggested by Barenblatt [2], the crack extension over which cohesive forces act would enter instead.

Fig. 5 shows a comparison between experimental data and equation (18). The tests were carried out on strips with a thickness of $^1/_{32}$ in., a width of $2b = 1^3/_8$ in., and a length of 10 in. Each data point represents the average of 3 measurements of average velocities over a length of about $^1/_2$ in. The crack velocities are small enough to be easily measured with the help of a stopwatch and optical comparator. The values of the shift factors ϕ_T for the test temperatures given in Fig. 5 are in good agreement with values determined by other tests [22, 24]. The actually observed relationship between strain ϵ_0, crack velocity v, and temperature T is seen to be well represented by equation (18), together with the material properties just discussed.

Implications for Nonsteady Crack Propagation

The crack-propagation equation for a strip (18) has been derived from equation (12) by giving the stress-intensity factor $I_n(a)$ and the stress $\hat{\sigma}_{yy}(t_j)$ appropriate values, which are independent of time and crack length in this case. It has been shown in [26], however, that equation (12) is also applicable when the stress-intensity factor is a function of crack length and the specimen is loaded by time-independent forces. The equation must then be viewed as a first-order nonlinear differential equation for the crack length $a(t)$.

The details of modifying equation (12) for stresses $\hat{\sigma}_{yy}(t)$, which change drastically during the time interval $t_j \leqslant t \leqslant t + \Delta t$ are given in [17]. Nevertheless, it is interesting to point out the implications of the current result for a time-dependent stress $\hat{\sigma}_{yy}(t)$. As an example we consider the strip geometry in Fig. 1 to be loaded by a strain ϵ_s which is applied suddenly at time $t = 0$ and held constant thereafter. To the extent that the assumption of a constant Poisson's ratio ν is admissible for the viscoelastic response in the near-glassy ($\nu_0 \cong 0.3$) and near-rubbery time domain ($\nu_\infty \cong 0.5$), the stress in the strip without crack is equal to

$$\hat{\sigma}_{yy}(t) = \frac{\epsilon_s}{1 - \nu^2} E_{\text{rel}}(t) \tag{22}$$

where $E_{\text{rel}}(t)$ stands for the tensile relaxation modulus.

Substituting (22) into (12) and restricting ourselves again to cracks with an initial length such that $\frac{a}{b} > 1.5$, we obtain the following expression for crack velocity as an implicit function of time t

$$\frac{\epsilon_s{}^2 E_{\text{rel}}{}^2(t)}{2(1 - \nu^2)} G\left(\frac{\Delta a}{v\phi_T}\right) \frac{T}{273} = \frac{S}{b} \tag{23}$$

The temperature effect has been included in this statement on the same basis as in the derivation of (18).

Since $E_{\text{rel}}(t)$ is a monotonically decreasing function with time, it follows, cf. Fig. 4, that the crack velocity v is also a decreasing function of time, provided

$$\epsilon_{0_{\max}} > \epsilon_s > \frac{E_r}{E_g} \epsilon_{0_{\min}} \tag{24}$$

The limit strains used to establish this inequality are defined by equations (19) and (20). In case the magnitude of ϵ_s is such that (24) is satisfied and $\epsilon_s < \epsilon_{0_{\min}}$, the crack will propagate for some distance and be arrested at time t^* after strain application. This time is implicitly given by

$$E_{\text{rel}}(t^*) = \frac{\epsilon_{0_{\min}}}{\epsilon_s} E_r \tag{25}$$

Since the relaxation modulus decays rapidly with time, crack arrest will occur within a short time unless ϵ_s is almost equal to the strain $\epsilon_{0_{\min}}$, below which steady crack propagation is impossible. It is of peripheral interest to note that, in the limit case of a perfectly elastic material ($E_r = E_g$; $\epsilon_{0_{\min}} = \epsilon_{0_{\max}}$), the inequality (24) merely indicates that $\epsilon_s = \epsilon_{0_{\min}}$, which corresponds to an unstable equilibrium state.

Concluding Remarks

It has been demonstrated that extending the Irwin analysis of tractions at the crack tip to linearly viscoelastic materials leads to a theory that is in reasonable agreement with experimental results on crack propagation in a strip. For a small crack growing in a large plate under constant external load, agreement between theory and experiment has also been demonstrated [26].

We therefore believe that this approach to crack propagation in viscoelastic materials provides a rational tool for the understanding of fracture in this class of material. It should be emphasized that brittle fracture is a limit case in this theory, and that it is sufficient to consider rate effects to arise solely from the viscoelastic constitutive behavior, leaving the fracture (surface) energy a rate-insensitive quantity and thus consistent with its meaning in brittle fracture.

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